Loop models, modular symmetry and duality in 2+1 dimensions

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Happy Birthday, Nick!

$SU(2)_3$, Taos, NM

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Order Parameter and Ginzburg-Landau Theory for the Fractional Quantum Hall Effect

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A new order parameter with a novel broken symmetry is proposed for the fractional quantum Hall effect, with the Laughlin state as the mean-field ground state. The classical Ginzburg-Landau theory of Girvin is derived microscopically from this starting point and exhibits all the phenomenology of the fractional quantum Hall effect.

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Theory of the half-filled Landau level

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A two-dimensional electron system in an external magnetic field, with Landau-level filling factor $\nu=\frac{1}{2}$, can be transformed to a mathematically equivalent system of fermions interacting with a Chern-Simons gauge field such that the average effective magnetic field acting on the fermions is zero. If one ignores fluctuations in the gauge field, this implies that for a system with no impurity scattering, there should be a *well-defined Fermi surface* for the fermions. When gauge fluctuations are taken into account, we find that there can be infrared divergent corrections to the quasiparticle propagator, which we interpret as a divergence in the effective mass m^*, whose form depends on the nature of the assumed electron-electron interaction $v(\mathbf{r})$. For long-range interactions that fall off slower than $1/r$ at large separation r, we find no infrared divergences; for short-range repulsive interactions, we find power-law divergences; while for Coulomb interactions, we find logarithmic corrections to m^*. Nevertheless, we argue that many features of the Fermi surface are likely to exist in all these cases. In the presence of a weak impurity-scattering potential, we predict a finite resistivity ρ_{xx} at low temperatures, whose value we can estimate. We compute an anomaly in surface acoustic wave propagation that agrees qualitatively with recent experiments. We also make predictions for the size of the energy gap in the fractional quantized Hall state at $\nu=p/(2p+1)$, where p is an integer. Finally, we discuss the implications of our picture for the electronic specific heat and various other physical properties at $\nu=\frac{1}{2}$, we discuss the generalization to other filling fractions with even denominators, and we discuss the overall phase diagram that results from combining our picture with previous theories that apply to the regime where impurity scattering is dominant.

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NONABELIONS IN THE FRACTIONAL QUANTUM HALL EFFECT

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Applications of conformal field theory to the theory of fractional quantum Hall systems are discussed. In particular, Laughlin's wave function and its cousins are interpreted as conformal blocks in certain rational conformal field theories. Using this point of view a hamiltonian is constructed for electrons for which the ground state is known exactly and whose quasihole excitations have nonabelian statistics; we term these objects "nonabelions". It is argued that universality classes of fractional quantum Hall systems can be characterized by the quantum numbers and statistics of their excitations. The relation between the order parameter in the fractional quantum Hall effect and the chiral algebra in rational conformal field theory is stressed, and new order parameters for several states are given.

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Beyond paired quantum Hall states: Parafermions and incompressible states in the first excited Landau level

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The Pfaffian quantum Hall states, which can be viewed as involving pairing either of spin-polarized electrons or of composite fermions, are generalized by finding the exact ground states of certain Hamiltonians with $k+1$-body interactions, for all integers $k\geqslant 1$. The remarkably simple wave functions of these states involve clusters of k particles, and are related to correlators of parafermion currents in two-dimensional conformal field theory. The $k=2$ case is the Pfaffian. For $k\geqslant 2$, the quasiparticle excitations of these systems are expected to possess non-Abelian statistics, like those of the Pfaffian. For $k=3$, these ground states have large overlaps with the ground states of the (two-body) Coulomb-interaction Hamiltonian for electrons in the first excited Landau level at total filling factors $\nu=2+3/5,2+2/5$. [S0163-1829(99)06911-8]

Motivation

- Dualities in CM and QFT
- Particle-Vortex duality and its applications to the Fractional Quantum Hall Effect
- Conjectured dualities, bosonization and fermionization
- Loop models: flux attachment, duality and periodicity
- Periodicity vs Fractional Spin
- Implications for Fractional Quantum Hall fluids

Dualities

- EM duality: E $\Longleftrightarrow$ B, electric charges $\Longleftrightarrow$ magnetic monopoles$\Rightarrow$Dirac quantization
- 2D Ising Model: Kramers-Wannier duality, high T $\Longleftrightarrow$ low T, order $\Longleftrightarrow$ disorder
- Duality of the 3D $\mathbb{Z}_2$ gauge theory $\Longleftrightarrow$ 3D Ising model, order $\Longleftrightarrow$ confinement
- Particle-Vortex duality: electric charge $\Longleftrightarrow$ vortex (magnetic charge)
- Mappings between phases of matter, most often between different theories
- Conjectured web of dualities between CFTs in 2+1 dimensions

Fractional Quantum Hall Effect

- Flux attachment for statistical transmutation
- Landau-Ginzburg theory (Read; Zhang, Hansson and Kivelson: Non-Relativistic abelian-Higgs model with a Chern-Simons term: Electrons are "composite" bosons coupled to *m* fluxes
- Fermionic flux attachment (Jain, López and Fradkin, Halperin, Lee and Read)
- FQH plateau: composite bosons condense; the excitations are anyonic vortices
- FQH insulator: bosons are uncondensed and the gapped excitations are fermions
- Universal phase diagram for the FQH states based on particle-vortex duality (Kivelson, Lee and Zhang) with "super-universal" transitions (superconductor-insulator transition)
- Apparent self-duality at the plateau transitions (I $\longleftrightarrow$ V) (Shimshoni, Sondhi and Shahar)

Bosonic Particle-Vortex Duality

- Particle-vortex duality of 3D XY model (Peskin, Stone, Halperin-Dasgupta)
  - U(1) broken symmetry phase ($m^2 < 0$); excitations: closed quantized vortex lines with long range interactions
  - U(1) unbroken symmetry phase ($m^2 > 0$); excitations: massive charged bosons, closed worldlines with short-range repulsive interactions
- Duality: particles $\Leftrightarrow$ vortices, high T $\Leftrightarrow$ low T, strong coupling $\Leftrightarrow$ weak coupling
- The 3D XY model as a loop model: lattice partition function sums over configurations of closed loops with short-range repulsive interactions
- Phase diagram for FQH fluids (Kivelson, Lee and Zhang) and plateau transitions
- 2+1-dimensional boson-boson complex scalar field theory mapping

$$|D(A)\phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4 \leftrightarrow |D(a)\varphi|^2 + m^2|\varphi|^2 - \lambda|\varphi|^4 + \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda$$

Web of Dualities

Recently conjectured dualities between *fixed points* (relativistic CFTs)

Fermionic particle-vortex duality (“QED_3”) (Son, Metlitski-Vishwanath):

$$i\bar{\psi}\not{D}(A)\psi - \frac{1}{8\pi}AdA \longleftrightarrow i\bar{\chi}\not{D}(a)\chi - \frac{1}{4\pi}adA - \frac{1}{8\pi}AdA$$

maps Time reversal to Charge conjugation (PH); B $\longleftrightarrow \mu$

Bosonization duality (Seiberg, Senthil, Wang and Witten):

$$i\bar{\psi}\not{D}(A)\psi - \frac{1}{8\pi}AdA \longleftrightarrow |D(a)\phi|^2 - |\phi|^4 + \frac{1}{4\pi}ada + \frac{1}{2\pi}adA$$

Our Strategy

Goldman and EF, Phys. Rev. B **97**, 195112 (2018); ibid. **98**, 165137 (2018)

- “Derive” this web of dualities using quantum loop models near criticality, but still in the gapped phases.

- These models are related to *modular invariant* models originally introduced by Kivelson and myself

- Modular invariance cannot be kept close to the CFT.

- “Fractional spin” breaks modular invariance, and gives rise to Dirac fermions, leading to loop model based “proofs” of the CFT duality web.

Quantum Loop Models and Duality

(EF and Kivelson 1996)

- Non-intersecting linked loops [J_μ] in 3D Euclidean space-time (with no spin) with exact particle-hole symmetry
- flux attachment with fractional statistics θ, long ranged interactions with coupling g, and short-range repulsion (to avoid crossings)
- The imaginary part of the action is given in terms of the loops linking number

$$Z[g,\theta] = \sum_{\{J_\mu\}\in\mathbb{Z}} \delta(\Delta_\mu J^\mu) e^{-S[J_\mu]}$$

$$S[J_\mu] = \frac{g^2}{2}\sum_{x,y} J_\mu(x) G_{\mu\nu}(x-y) J_\nu(y) + i\theta \sum_{x,y} J_\mu(x) K_{\mu\nu}(x-y) J_\nu(y)$$

long ranged interactions

linking number $=\theta\Phi[J]$

$$G_{\mu\nu}(p) = \frac{1}{\sqrt{p^2}}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right), \; K_{\mu\nu}(p) = i\epsilon_{\mu\nu\lambda}\frac{p_\lambda}{p^2}$$

Field theory picture: 2+1 D complex scalar field coupled to 3+1 D Maxwell field with a θ term

$$\mathcal{L} = |D_\mu\phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4 - \frac{1}{4g^2} f_{\mu\nu}\frac{1}{\sqrt{-\partial^2}} f_{\mu\nu} + \frac{k}{4\pi}\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

Self-Duality and Modular Invariance

Modular parameter: $$\tau = \frac{\theta}{\pi} + i\frac{g^2}{2\pi}$$

- The partition functions of loop models regularized without self-linking (fractional spin) have the symmetries
- $\mathcal{S}$: duality: Z[τ]=Z[-1/τ], and $\mathcal{T}$: Periodicity: Z[τ]=Z[τ+1]
- $\mathcal{S}$ and $\mathcal{T}$ generate the modular group PSL(2,$\mathbb{Z}$)
- The partition function is self dual at the fixed points of the modular group
- Two types of PSL(2,$\mathbb{Z}$) fixed points: “bosonic” and “fermionic”
- FK showed that the finite modular fixed points are quantum critical points with $\sigma_{xx} \neq 0$ and $\sigma_{xy} = 0$
- The predicted conductivities are different in the FK loop models and the relativistic web of dualities

The Role of Fractional Spin

- The linking number of two separate loops l_1 and l_2 is

$$\Phi[J = \ell_1 + \ell_2] = 2 \times (\text{Linking number of } \ell_1 \text{ with } \ell_2) + W[\ell_1] + W[\ell_2]$$

"Writhe." Associated with self linking. Not necessarily a topological invariant.

- Witten: point-split the loops into ribbons so that the writhe is a frame-dependent topological invariant $W[l] = SL[l]$ = integer. Only consistent deep in the topological phase, not as the critical point is approached.

- Polyakov: no-point splitting and $W[l] = SL[l] - T[l]$ (writhe = self-linking - twist)

$$T[\ell] = \frac{1}{2\pi}\int_0^L ds \int_0^1 du\, \boldsymbol{e}\cdot\partial_s \boldsymbol{e}\times\partial_u \boldsymbol{e}$$

$T[l]$ is a Berry phase (fractional spin) and $\boldsymbol{e}$ is the tangent vector to the loop. The twist $T[l]$ is not quantized and depends on the metric.

Fractional Spin:
Periodicity Lost, 3D Bosonization Regained

- *T*[l] is not quantized. Means Duality $\mathcal{S}$ remains a symmetry, but periodicity $\mathcal{T}$ is lost

- Polyakov: fractional spin leads to the (IR) duality between a complex massive scalar with CS at k = 1 and a massive Dirac *spinor* (with a parity anomaly)

- Loop model representation

$$Z_{\mathrm{fermion}} = \det[i\partial\!\!\!/ - M] = \int \mathcal{D}J\, \delta(\partial_\mu J^\mu)\, e^{-|m|L[J] - i\mathrm{sign}(M)\pi\Phi[J]}$$

L[*J*]: length of loop, *Φ*[*J*]: linking number (including the spin factor)

For general statistical angle θ we have the loop model

$$Z = \int \mathcal{D}J\, \delta(\partial_\mu J^\mu)\, e^{-|m|L[J] + i\theta\Phi[J]}$$

Fractional Spin: Periodicity Lost, 3D Bosonization Regained

- Can we use this to "derive" the web of dualities? Yes!

- First step: We introduce background fields to the boson side of Polyakov's duality in the unbroken phase

$$\mathcal{L}_B = |D[a]\phi|^2 - m_0^2|\phi|^2 - |\phi|^4 + \tfrac{1}{4\pi}ada + \tfrac{1}{2\pi}adA$$

Exact rewriting as loop model coupled to gauge fields

$$Z[A] = \int \mathcal{D}J\mathcal{D}a\,\delta(\partial_\mu J^\mu)\,e^{-|m|L[J]+iS[J,a,A]}$$

$$S[J,a,A] = \int d^3x\left[J(a-A) + \frac{1}{4\pi}ada - \frac{1}{4\pi}AdA + \dots\right]$$

Integrating-out *a* results in a term involving the linking number and the spin factor

$$-\pi\,\Phi[J] + \int d^3x\left[JA - \frac{1}{4\pi}AdA\right]$$

$$\mathcal{L}_F = \bar{\Psi}(i\not{D}[A] - M)\Psi - \frac{1}{8\pi} AdA$$

with M<0

Loop model representation

$$Z_{\text{fermion}}[A; M < 0]\, e^{-i\,\text{CS}[A]/2} = \int \mathcal{D}J\, \delta(\partial_\mu J^\mu)\, e^{-|m|L[J] + iS_{\text{fermion}}[J,A;M<0]} e^{-i\,\text{CS}[A]/2}$$

$$S_{\text{fermion}}[J, A; M < 0] = \int d^3x \left(JA - \frac{1}{8\pi} AdA \right) - \pi\Phi[J]$$

The bosonization identity in the phase with broken time reversal, M>0, is obtained by a particle vortex duality in the bosonic theory

Loop Models: Tools for Deriving Dualities

1. Start with a proposed duality and write down boson loop models for each theory using Polyakov's duality.

2. Use path integral manipulations to equate the two loop model partition functions.

3. Match both sides of the critical point using bosonic particle-vortex duality. Relates superfluid of particles to insulator of vortices.

4. The dualities are IR identities

5. In the bosonic theories the short-distance repulsion between loops become the ϕ^4 coupling, which in the massless limit flow in the IR into the WF fixed point

Example: Fermion particle-vortex duality

Use loop models to derive the duality between free Dirac fermion and QED$_3$ with (quantized) Chern-Simons terms

$$i\bar{\Psi}\not{D}[A]\Psi - \tfrac{1}{8\pi}AdA \leftrightarrow i\bar{\psi}\not{D}[a]\psi + \tfrac{1}{8\pi}ada - \tfrac{1}{2\pi}adb + \tfrac{2}{4\pi}bdb - \tfrac{1}{2\pi}bdA$$

$$-M\bar{\Psi}\Psi, \quad M < 0 \qquad\qquad -M'\bar{\psi}\psi, \quad M' > 0$$

$$\int d^3x\, J_\mu A^\mu + \pi\Phi[J] \longleftarrow -\pi\Phi[J] + \int d^3x \left[J_\mu a^\mu - \frac{1}{2\pi}adb + \frac{2}{4\pi}bdb - \frac{1}{2\pi}bdA\right]$$

Integrate out a, b

$$Z_{\mathrm{F}}[A; M < 0] = Z_{\mathrm{QED}_3}[A; M' > 0], Z_{\mathrm{F}}[A; M > 0] = Z_{\mathrm{QED}_3}[A; M' < 0]$$

- Case for opposite mass signs (QH phase) follows from the same logic
- Current mapping also natural upon integrating out b:

$$\bar{\Psi}\gamma^\mu\Psi \longleftrightarrow \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\partial^\nu a^\lambda$$ (derived earlier by EF and F. Schaposnik, 1994)

Compressible “FQH” states and Duality

- The Jain sequences of FQH states ν(p, n)=p/(2np±1) converge to ν=1/(2n) where the FQH gap vanishes $\mapsto$ Halperin, Lee, Read theory of a composite Fermi liquid

- This theory had great successes. It also has problems: in the simplest case, n=1, $\nu \mapsto$ 1/2 and PH symmetry is expected (for large B).

- HLR is not compatible with PH (DH Lee)

- The “Fermi liquid” is a “Non-Fermi liquid”

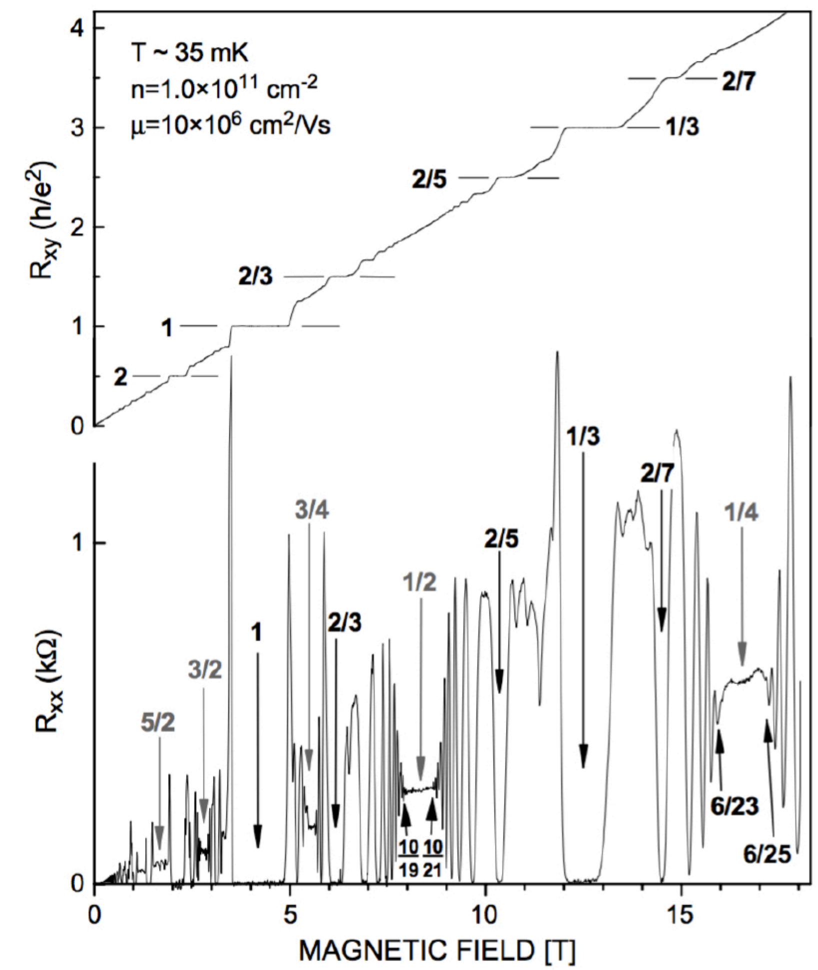
- Son proposed a relativistic version of HLR which satisfies PH

- At finite μ (Fermi surface!) this is still a “non-Fermi liquid”

- What about the ν=1/2n compressible states where PH should not hold?

1/2n Compressible States

- Compressible states with ν=1/2n are predicted by the Jain sequences
- They are seen in experiment
- PH does not hold for general n
- Reflection symmetry of the I-V curves at plateau transitions
- Interpreted as evidence of particle-vortex duality (Shahar, Shimshoni, Sondhi)
- For ν=1/2 PH symmetry relates ρ_{xx} to σ_{xx} and $\nu \longleftrightarrow 1-\nu$



from [Stern, Ann. Phys. (2008)],
data from W. Pan (Sandia)

Symmetries at 1/2n Compressible States

- The same reflection symmetry is seen at ν=1/4, locus of ν=1/3 $\mapsto$ 0 transition (where $\nu \nleftrightarrow$ 1-ν), with ρ_{xy}=-3e^2/h

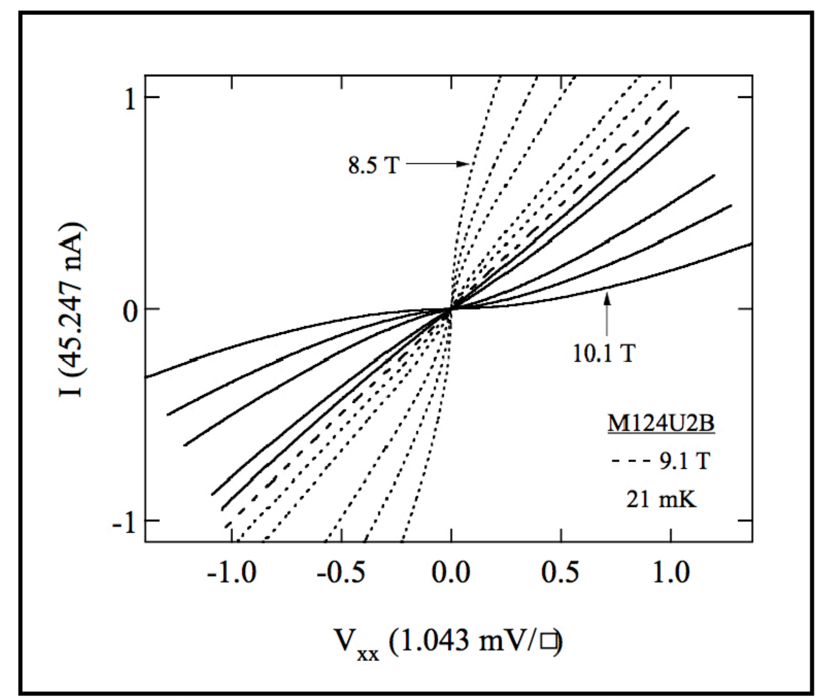
- This is not PH symmetry!

- For ν=1/2n the symmetry is between the Jain states at ν=p/(2np+1) and ν'=(1+1)/(2n(1+p)-1), both converging to 1/2n

- For reflection symmetry to hold the HLR composite fermions must have σ_{xy}=-e^2/2h

- Flux attachment breaks PH and reflection explicitly

- Same problems in Son's theory which needs to be modified to treat ν and ν' equitably



"Charge-flux" symmetry at $\nu = 1/2n$.
[Shahar *et al.*, Science (1996)].

Reflection symmetry at ν=1/2n

$$\mathcal{L}_{1/2n} = i\bar{\psi}\not{D}_a\psi - \frac{1}{4\pi}\left(\frac{1}{2} - \frac{1}{2n}\right) ada + \frac{1}{2\pi}\frac{1}{2n} adA + \frac{1}{2n}\frac{1}{4\pi} AdA$$

- *A* is the external gauge field of strength *B*, and *a* is the Chern-Simons field (flux attachment)

electron filling: $$\nu = \frac{2\pi}{B}\left\langle \frac{\delta \mathcal{L}_{\nu=1/2n}}{\delta A_0} \right\rangle = \frac{1}{2n}\left(1 + \frac{b_*}{B}\right)$$

$$b_* = 0 \Rightarrow \nu = \frac{1}{2n}$$

Composite fermion ψ FS set by a_0: $$\rho_\psi = \frac{1}{2\pi}\left(\frac{1}{2} - \frac{1}{2n}\right) b_* - \frac{1}{2\pi}\frac{B}{2n}$$

$$\nu_\psi = 2\pi\frac{\rho_\psi}{b_*} = \frac{1}{2} + \frac{\nu}{1-2n\nu} \qquad \nu_\psi = p + \frac{1}{2} \Rightarrow \nu = \frac{p}{2np+1}$$

$$\nu_\psi \to -\nu_\psi \Leftrightarrow \nu = \frac{p}{2np+1} \to \frac{1+p}{2n(1+p)-1}$$

Reflection symmetry and boson self-duality

$$\mathcal{L}_{1/2n} \leftrightarrow |D_{g-A}\phi|^2 - |\phi|^4 + \frac{1}{4\pi}\frac{1}{2n-1} gdg$$
$$\leftrightarrow |D_h\varphi|^2 - |\varphi|^4 - \frac{2n-1}{4\pi} hdh + \frac{1}{2\pi} hdA$$

- First line: fermion-boson duality
- Second line: boson-vortex duality
- relates ν_ϕ to - 1/ν_φ
- ν=1/2n $\Leftrightarrow$ ν_ϕ = - ν_φ=1
- Reflection related filling fractions ν_ϕ (ν) = - ν_φ(ν')
- Reflection symmetry is boson-vortex exchange
- Reflection symmetry at ν=1/2n $\Leftrightarrow$ boson self-duality!

Conclusions

- Loop models offer a bridge to the "derivation" of the web of dualities for relativistic theories
- The loop models are always interacting and scalar fields are never free
- Fractional spin plays a key role
- It is always possible to find a dual theory
- Periodicity of flux attachment, and SL(2,$\mathbb{Z}$), is not a symmetry for relativistic theories
- New insights from duality (and self-duality) on the ν=1/2n states of the FQH fluids